Indefinite Integration of the Gamma Integral and Related Statistical Applications

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Abstract

Liouville's theorem proves that certain integrals cannot be evaluated with elementary functions. It demonstrates why the gamma, exponential and Gaussian integrals lack antiderivatives. However, by applying the "h" factorization method, the author presents an analytical solution to the antiderivative of the gamma integral. This solution applies to all integrals that can be transformed into a gamma integral, including the exponential and Gaussian integrals. The author further provides a thorough discussion of the algebraic properties of the "h" function. The major contribution to statistical science is that "h" can serve as the most fundamental function which unifies many cumulative distribution functions, such as the gamma function, the exponential integral function, the error function, the beta function, the hypergeometric function, and the Marcum Q-function, and the truncated normal distribution. The closed-form expression of the moment-generating function for the truncated normal distribution can be also derived as an "h" function.

Keywords: Exponential integral Gaussian integral Cumulative distribution function Liouville's theorem

1 Introduction

Most beginning students of statistics are bewildered by the many statistical tables in their textbook appendices. They are taught that these tables are indispensable for evaluating the probability of certain distributions, such as a Z distribution, F distribution, or χ^2 distribution. As those students become more knowledgeable, they realize that the true reason for using statistical tables stems from the lack of antiderivatives for certain integral functions such as the gamma, exponential, and Gaussian integrals (Risch, 1969, 1970; Rosenlicht, 1972). Historically, the three integrals were proposed in Euler (1729), Mascheronio (1790), and de Moivre (1733). The first tables, however, were not available until Pearson (1922), Glaisher (1870), and Kramp (1799), respectively created them. Since the cumulative distribution functions of many distributions are related to the three integrals, the use of prepared tables was the only way to evaluate a probability before the computer era.

In modern calculus, the indefinite integral is called "antiderivative" because it is an inverse operator of derivative. According to the fundamental theorem of calculus (Stewart, 2003, pp.284-290), all continuous functions have antiderivatives, but only some of them possess antiderivatives that can be expressed by elementary functions. Mathematicians today explain this result using the differential Galois theory, for which Liouville's theorem provides the basic argument: if an elementary antiderivative exists, it must be in the form of an elementary function constructed via simple arithmetic operations in a finite number of steps (Conrad, 2005; Fitt & Hoare, 1993; Kasper, 1980). These elementary functions include polynomial, trigonometric, exponential, and logarithmic functions; the simple operations comprise addition, subtraction, multiplication, division, and root extractions.

Because they are based on Liouville's theorem, many indefinite integrals are believed to be unevaluable in finite terms of elementary functions, including the gamma, exponential, and Gaussian integrals. Nevertheless, a logical inconsistency exists between the definition of elementary functions and the concept of finite terms since trigonometric, exponential, and logarithmic functions by definition are all infinite series. They become "closed-formed" because mathematicians define them as elementary functions even though they are infinite series in nature (Chow, 1999). In fact, many irrational numbers, which cannot be expressed by a simple fraction, can be regarded as infinite series (Manning, 1906). What Liouville's theorem proves is contingent upon the definition of elementary functions, and the acceptance of this definition largely prevents any efforts in discovering the antiderivatives of those "unsolvable" integrals.

Are the gamma, Gaussian, and exponential integrals truly unsolvable? If we set aside the requirement of the elementary function, the antiderivative of the three integrals can be expressed as an infinite series. To see why this is so, we first define a general form of the gamma integral

$$g\left(s,c,u\right) \equiv \int u^{s-1}\exp(cu)du,$$

where s and c can be any real numbers. If we assign particular values to both parameters, the Gaussian and exponential integrals can be viewed as a special case of the gamma integral. For example, when s is a non-positive integer, it is an exponential integral of $(1-s)$th order. Specifically, it results in the first-order exponential integral if s is set to 0

$$\int \frac{\exp(-u)}{u}du = g\left(0,-1,u\right).$$

Moreover, by substitution of variables, we can derive an identity of the Gaussian integral as a gamma integral with $s = 1/2$ and $c = -1$

$$\int \exp(-u^2)du = \frac{1}{2}g\left(\frac{1}{2},-1,u^2\right).$$

Therefore, all three integrals can be generalized into a gamma integral $g\left(s,c,u\right)$. Let $c = -1$ and set the lower and upper limits of the integral as 0 and ∞. We will derive the gamma function $\Gamma\left(s\right)$. Solving a gamma integral is straightforward when we use the Taylor expansion and then integrate the series term by term

$$g(s,-1,u)=\frac{u^{s}}{0!\,(s)}-\frac{u^{s+1}}{1!\,(s+1)}+\frac{u^{s+2}}{2!\,(s+2)}-\frac{u^{s+3}}{3!\,(s+3)}+\cdots. \qquad (1)$$

This solution is an infinite series and usually applied to evaluation of the lower incomplete gamma function, which can be specified as

$$\gamma\left(s,u\right)=\int_{0}^{u}x^{s-1}\exp(-x)dx.$$

We can also express the lower incomplete gamma function as an infinite series of Kummer's confluent hypergeometric function

$$\gamma\left(s,u\right)=\frac{u^{s}}{s}M\left(s,s+1,-u\right),$$

where the Kummer's confluent hypergeometric function is defined as

$$M\left(a,b,u\right)=1+\frac{a}{b}u+\frac{a\left(a+1\right)}{b\left(b+1\right)}\frac{u^{2}}{2!}+\frac{a\left(a+1\right)\left(a+2\right)}{b\left(b+1\right)\left(b+2\right)}\frac{u^{3}}{3!}+\cdots.$$

Respecifying $\gamma\left(s,u\right)$ as an indefinite integral with variable u, we can regard the infinite series of Kummer's confluent hypergeometric function as a solution to the indefinite integration of the gamma integral

$$\int u^{s-1}\exp(-u)du=\frac{u^{s}}{s}M\left(s,s+1,-u\right). \qquad (2)$$

A question comes up immediately concerning why (1) or (2) was not widely recognized as a general solution for the gamma integral. Two related accounts can shed some light on this question. First, while both $g(s,-1,u)$ and $M\left(s,s+1,-u\right)$ have an infinite radius of convergence, they require a massive computational capacity to achieve a minimum level of accuracy when u is large. Second, especially for the gamma function, since the upper limit of the integral is infinity, all the terms of $g(s,-1,u)$ and $M\left(s,s+1,-u\right)$, except the constant of M, approach infinity and result in an indefinite outcome (Borwein & Borwein, 2011). Apparently, we cannot use (1) or (2) to evaluate a gamma integral when it is sepcified as an improper integral.

The essence of the first problem is numerical precision. A previous study shows that we need at least 60 digits of precision to get one digit of accuracy in evaluating $M(a, b, z)$, where $a = b$ and $z = 0 + 140i$ (Nardin *et al.*, 1992, p.194). Many computational methods are developed to solve this problem, especially when a and b are small but z is larger than 50 (Muller, 2001, p.50). However, a more difficult and fundamental problem remains unsolved, that is, how can we evaluate the asymptotic properties of the gamma integral when its upper limit approaches infinity? If we can answer this question, theoretical ground would exist to recognize (1) or (2) as a solution to the gamma integral.

In this paper, the author applies a different factorization method to investigate the gamma integral with a new function "h", which can be expressed in terms of Kummer's confluent hypergeometric function

$$h_s^c \equiv \frac{1 - M(1, s+1, -c)}{c}. \tag{3}$$

Through the analysis of the "h" function, the author found that "h" directly links to the gamma function, and its asymptotic property is well-defined when the argument c approaches negative infinity. This finding not only gives a general definition of the gamma function that extends the concept of the factorial function to the non-integer domain, but also completes the solution of the gamma integral, which can be further applied to the cumulative distribution functions of many important distributions. Moreover, the closed-form expression of the moment-generating function for the truncated normal distribution can be also derived as an "h" function. The overall finding is a significant contribution to general statistical science, and its application radically changes current practices in calculating probability.

2 “h” factorization of the Gamma integral

According to its relationship with the confluent hypergeometric function in (3), “h” can be specified as

$$h_s^c = \frac{(-c)^0}{(s+1)} + \frac{(-c)^1}{(s+1)(s+2)} + \cdots + \frac{(-c)^n}{\prod\limits_{i=0}^{n}(s+1+i)}, \; n \to \infty. \tag{4}$$

We factorize the gamma integral into a product of the power function, the exponential function, and the “h” function

$$g(s,c,u) = u^s \exp(cu) h_{s-1}^{cu}. \tag{5}$$

Every “h” function is an infinite series, in which the base parameter s and the power parameter c determine its functional value. Base parameter s defines the starting number of the factorial term in the denominator. Power parameter c defines the negative power term in the numerator. While s and c can be any real number, we temporarily exclude the case where s is a negative integer and will return to it in the next section.

Given a constant c, h_s^c is a convergent series. As the number of the expansion terms increases, the factorial denominator will eventually overpower the numerator, and the incremental value becomes infinitesimal. Furthermore, h_s^c will approach 0 as s departs from 0, either in the positive or negative direction, because the absolute value of the denominator becomes larger and larger. Computationally, using an nth-order expansion to calculate h_s^c possesses an error of $O(c^{n+1})$ as c approaches 0

$$h_s^c = \sum_{i=0}^{n} \frac{(-c)^i}{\prod\limits_{j=0}^{i}(s+1+j)} + O\left(c^{n+1}\right).$$

When the integration interval is set from 0 to ∞, the gamma integral $g(s+1,-1,u)$ becomes $\Gamma(s+1)$. If s is a nonnegative integer, it defines the factorial function $s!$. If s is not an integer, due to its recursive regularity, all gamma functions can be calculated as a function of $\Gamma\left(\operatorname{mod}(s,1)\right)$, where $0 < \operatorname{mod}(s,1) < 1$.

$$\Gamma(s+1) = \begin{cases} \Gamma\left(\operatorname{mod}(s,1)\right) \prod\limits_{i=0}^{\lfloor s \rfloor} (s-i), & \text{if } s > -1 \text{ \& } s \notin \mathbb{Z}_0^+; \\ \Gamma\left(\operatorname{mod}(s,1)\right) \prod\limits_{i=0}^{|\lceil s \rceil|-1} (s+1+i)^{-1}, & \text{if } s < -1 \text{ \& } s \notin \mathbb{Z}^-; \\ s!, & \text{if } s \in \mathbb{Z}_0^+. \end{cases} \tag{6}$$

For any non-integer s, the remainder for one, $\operatorname{mod}(s,1)$, is bounded within the interval $(0,1)$. Thus, all gamma functions can be calculated if $\Gamma\left(\operatorname{mod}(s,1)\right)$ is known. Given that the "h" function is derived from factorizing the gamma integral, the investigation of "h" will focus on the same integral limit as $\Gamma\left(\operatorname{mod}(s,1)\right)$.

An identity of the "h" function can be respecified from (5) (See Appendix A.1 for the proof.)

$$h_{k-r}^{c} = \exp(-c) \sum_{i=0}^{\infty} \frac{c^i}{i!\,(k+1+i-r)}. \tag{7}$$

Here, k is a non-negative integer and $0 < r < 1$. Performing long division for $1/[(k+1+i)-r]$, we can derive

$$h_{k-r}^{c} = \exp(-c) \sum_{i=0}^{\infty} \left\{ \frac{c^i}{i!} \left[\sum_{j=0}^{\infty} \frac{r^j}{(k+i+1)^{j+1}} \right] \right\}. \tag{8}$$

If we rearrange (8) and sum all the terms by i given j, we derive

$$h_{k-r}^{c} = \frac{\exp(-c)}{c^{k+1}} \sum_{i=0}^{\infty} r^i I^{(i)}(c,k), \tag{9}$$

where

$$I^{(0)}(c,k) = \int c^k \exp(c)\, dc,$$

and

$$I^{(i)}\left(c,k\right)=\int\frac{\overset{(i)}{\cdots}\int\frac{I^{(0)}(c,k)}{c}dc}{c}dc.$$

Appendix A.2 shows the proof that $I^{(n)}\left(c,k\right)$ can be further simplified as

$$I^{(n)}\left(c,k\right)=\sum_{i=0}^{\infty}\left\{\frac{(-1)^{n-1+i}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{\prod_{j=0}^{i}\left(k+1+j\right)}\right]I^{(0)}\left(c,k+i\right)\right\}. \quad (10)$$

With the proof in Appendix A.3, h_{k-r}^{c} can be reduced to

$$h_{k-r}^{c}=c^{-k-1}\exp(-c)I^{(0)}\left(c,k\right)+c^{-k-1}\exp(-c)r\sum_{i=0}^{\infty}\frac{(-1)^{i}I^{(0)}\left(c,k+i\right)}{\prod_{j=0}^{i}\left(k+1+j-r\right)}. \quad (11)$$

As mentioned earlier, we are interested in the gamma integral where the base parameter s is bounded within the interval $(0,1)$. Applying the "h" factorization as shown in (5), we derive $-1<k-r<0$ and $k=0$, given $0<r<1$. Thus,

$$\begin{aligned}h_{-r}^{c}=c^{-1}\exp(-c)\Bigg\{&(\exp(c)-1)+r\left(\frac{\exp(c)-1}{0!}\right)\left(\frac{1}{1-r}\right) \quad (12)\\&+r\left(\frac{c\cdot\exp(c)-\exp(c)+1}{1!}\right)\left(\frac{1}{2-r}-\frac{1}{1-r}\right)\\&+r\left(\frac{c^{2}\cdot\exp(c)-2c\cdot\exp(c)+2!\cdot\exp(c)-2!}{2!}\right)\left(\frac{1}{3-r}-\frac{2}{2-r}+\frac{1}{1-r}\right)+\cdots\Bigg\}.\end{aligned}$$

Since $\Gamma\left(-r+1\right)=c^{-r+1}\exp(-c)h_{-r}^{-c}\Big|_{c\to\infty}$, we can respecify (12) and replace the power-term parameter c with $-c$

$$h_{-r}^{-c}=c^{-1}\exp(c)\left[1-\exp(-c)\right]+c^{-1}\exp(c)r\sum_{i=0}^{n}w_{i}\beta_{i}, \quad (13)$$

where $w_i = 1 - \sum_{j=0}^{i} \frac{c^j \exp(-c)}{j!}$, $\beta_i = \frac{i!}{\prod_{j=0}^{i} (j+1-r)}$, and $n \to \infty$.

As demonstrated in Appendix A.4, we can reduce h_{-r}^{-c} to its simplest form, which directly links to the gamma function of $\Gamma(-r)$

$$h_{-r}^{-c} = -c^{-1} - rc^{r-1}\exp(c)\Gamma(-r). \tag{14}$$

Therefore, "h" explains the recursive rule of the gamma function

$$\begin{aligned} \Gamma(-r+1) &= c^{-r+1}\exp(-c)h_{-r}^{-c}|_{c\to\infty} \\ &= -r\Gamma(-r). \end{aligned} \tag{15}$$

This finding not only defines the algebraic meaning of the "h" function by connecting to the gamma function, but also generalizes the factorial function to non-integer cases. As is evident in (15), we can explicitly define $\Gamma(s)$ as a function of $h_{s-1}^{-c}|_{c\to\infty}$, where $0 < s < 1$. Hence, all gamma functions defined in (6) can be expressed as an "h" function for all real arguments, except for the case of non-positive integers.

Since $r \in (0,1)$, we can assume $r = q/p$, where $0 < q < p$, and derive an identity expression of $\Gamma(-r+1)$

$$\Gamma(-r+1) = \frac{p}{p-q}\int_0^\infty \exp(-x^{\frac{p}{p-q}})dx.$$

[Figure 1 here.]

Plotting the integrand function $\exp\left(-x^{p/(p-q)}\right)$ as Figure 1 shows, we know that the error is trivial for calculating the gamma function by replacing the upper limit ∞ with a considerably larger number, noted as "C" herefater. Consider the case that has the largest error, where $p/(p-q) \to 1$, the errors can be evaluated by

$$\int_C^\infty \exp(-x)dx = -\exp(-C).$$

The result indicates the errors are $-2.06e-09$, $-1.93e-22$, and $-3.72e-44$ for $C = 20$, 50, and 100, respectively. Given (15), we can always evaluate the gamma function with arbitary precision by employing the "h" factorization method.

For computational purposes, we need to set a proper value for C without causing a significant error when evaluating the gamma function. Doing so helps us understand the algebraic properties of the "h" function. As (13) shows, the core of the "h" function, $\lim_{n\to\infty} \sum_{i=0}^{n} w_i \beta_i$, is a monotonically decreasing series, that is composed of two sets of parameters: w_i and β_i. The former serves as a weighting factor, the ratio of the gamma integral with an upper limit c to the same integral with an infinite upper limit. The latter is a function of ith order forward difference, starting from $1/(1-r)$ and ending at $1/(i+1-r)$. The upper limit c is supposed to approach infinity, as is the number of expansion terms n.

The value of the weighting factor w_i is bounded within the interval $(0, 1)$ and can be expressed as follows:

$$w_i = \frac{\int_0^c x^i \exp(-x) dx}{\int_0^\infty x^i \exp(-x) dx}. \tag{16}$$

Theoretically, c should be set as ∞, which makes all w_i become one. Meanwhile, as the number of expansion terms n increases, the value of the difference function will approach a limit determined by $\Gamma(1-r)$ and $n^{-(1-r)}$.

$$\begin{aligned}\lim_{n\to\infty} \beta_n &= \lim_{n\to\infty} \frac{n!}{(1-r)(2-r)\cdots(n+1-r)} \\ &\to \frac{\Gamma(1-r)}{n^{1-r}}.\end{aligned}$$

Notice that β_i is monotonically decreasing to the convergence value,

$$\beta_0 > \beta_1 > \cdots > \beta_n \to \frac{\Gamma(1-r)}{n^{1-r}}$$

and thus, the sum of the infinite series $\lim_{n\to\infty} \sum_{i=0}^{n} w_i \beta_i$ will approach infinity

$$\lim_{n\to\infty} \sum_{i=1}^{n} w_i \beta_i > \lim_{n\to\infty} \frac{n\Gamma(1-r)}{n^{1-r}} \to \infty. \tag{17}$$

This result obviously deviates from our conclusion as shown in Appendix A.4

$$\lim_{n\to\infty} \sum_{i=0}^{n} w_i \beta_i = -c^r \Gamma(-r) - \frac{1}{r}.$$

The two conflicting results make clear that the nature of the problem is related to the invalid asymptotic analysis when we treat the upper limit of the gamma integral as infinity.

What is wrong with treating c as infinity? Negligence of the relative power of c and n to asymptotic infinity drives an incorrect conclusion in (17), that all of the weighting factor w_i is one, and thus $\lim_{n\to\infty} \sum_{i=1}^{n} w_i \beta_i$ approaches infinity. As (16) indicates, for any given value of c, we can always find a larger number N that makes w_{N+j}, where $j \in \mathbb{Z}_0^+$, approach 0. By taking a derivative, we know that the integrand function $x^N \exp(-x)$ has the maximum value at $x = N$. Thus, the integral value at $[0, c]$ as a ratio of the same integral value at the overall domain $[0, \infty]$ will approach 0 if $c \ll N$. This means, w_i cannot always be one because N will eventually overpower c in creating the weighting factor.

This conclusion explains why the infinite series based on the Taylor series (1) or Kummer's confluent hypergeometric function (2) was not widely recognized as a solution to the gamma integral. Employing the "h" factorization method, we can actually establish the identity between "h" and the gamma function, with an understanding that the infinite upper limit c should be regarded as a finite number. Once we realize the finite property of c, we are on firm ground to understand the "h" factorization method presented in (5) as a general solution to the gamma integral.

3 Basic properties of the "h" function

In the previous section, the discussion of the "h" function was restricted to real numbers of the base parameter s, except negative integers. This restriction can be relaxed with the following definition of "h":

$$h_s^c = \begin{cases} \sum\limits_{i=0}^{\infty} \frac{(-c)^i}{\prod\limits_{j=0}^{i}(s+1+j)}, & \text{if } s \notin \mathbb{Z}^-; \\ \exp(-c)\left(\log|c| + \sum\limits_{s=0}^{\infty} h_s^{-c}\right), & \text{if } s = -1; \\ \sum\limits_{i=0}^{-s-2} \frac{(-c)^i}{\prod\limits_{j=0}^{i}(s+1+j)} + \frac{(-c)^{-s-1}}{\prod\limits_{j=0}^{-s-2}(s+1+j)} h_{-1}^c, & \text{if } s \in \mathbb{Z}^- \ \&\ s \neq -1. \end{cases} \tag{18}$$

When s is not a negative integer, "h" is regarded as the core function of the solution to the indefinite gamma integral. Its asymptotic property at $c \to -\infty$ directly links to the gamma function. When s is a negative integer, "h" is related to the exponential integral, but only in the sense of definite integral. Specifically, we can apply h_s^c to an nth-order exponential integral with a definite lower and upper limit a and b.

$$\int_a^b \frac{\exp(-x)}{x^n} dx = x^{-n+1} \exp(-x) h_{-n}^{-x} \Big|_a^b$$

The above formula is only meaningful if a and b are positive finite numbers. We can explain why that is the case by discussing the infinite series that defines h_{-1}^c. As (18) makes evident, h_s^c can be reduced to a function of h_{-1}^c when s belongs to negative integers. By definition, h_{-1}^c is a two-term product, in which the latter term is the sum of $\log|c|$ and an infinite number h_s^{-c}, beginning at $s = 0$ and extending to $s \to \infty$. Since all of h_s^{-c} are positive when $s \geq 0$, the value of h_{-1}^c is contingent on the actual number of terms, noted as "t" hereafter, used to compute $\sum\limits_{s=0}^{\infty} h_s^{-c}$. As a result, its numerical magnitude can not be interpreted directly as the antiderivative value in absolute terms.

When the upper limit is infinity, we encounter a problem in evaluating the asymptotic property of $h_{-1}^{-c}|_{c\to\infty}$. Fortunately, the finite property of the infinite upper limit applies to all of the "h" functions used in calculating $h_{-1}^{-c}|_{c\to\infty}$. Therefore, the error of replacing the infinite upper limit with a considerably larger number C is trivial.

We refer to two distinct sources of error in applying the "h" function to evaluate the exponential integral. The first is the actual number of terms t in the computation of h_{-1}^{c}. This is a pure numerical problem, and it is similar to the number of terms from a Taylor series we need to compute an exponential number. The second source of error comes from the evaluation of the "h" function when the power parameter c approaches infinity. A discussion of these two kinds of error can be found in section 4.1 and 4.3.

The functional form of "h" does somewhat resembles exp, but "h" is not a Taylor series since the denominator is not a factorial term starting from 0!. If we set $s = -1$, all the denominator terms become 0 and "h" can not be defined by (4). Despite their difference, we can easily find certain identities between "h" and exp when s is a non-negative integer, such as $h_0^c = [1 - \exp(-c)]/c$.

[Figure 2 here.]

Unlike the exponential function, the power parameter c of the "h" function takes a negative value in composing the series, and hence, the shape of $y = h_s^c$ is downward-sloping and decreases to 0^+ as c increases to infinity, which is similar to the shape of the exponential function $\exp(-x)$. In addition, the intercept value of the "h" function is $1/(s+1)$, and this also differs from the case of the exponential function, which is always one.

Figure 2 presents functional plots of $\exp(-x)$ and h_s^x with six different base parameters. Despite the similar shape, $\exp(-x)$ intersects with all of the "h" functions. This clearly demonstrates that $\exp(-x)$ and h_s^x are two distinct functions. When $s > -1$, h_s^x is a monotonically decreasing function from $x = -\infty$ to $x = \infty$. When $s < -1$, the monotomic increase or decrease of h_s^x depends on the odd or even number of $\lfloor s+1 \rfloor$. If $\lfloor s+1 \rfloor$ is an odd number, h_s^x is a monotonically increasing function, such as $h_{-1.5}^x$ or $h_{-3.5}^x$; otherwise, h_s^x is a monotonically decreasing function, such as $h_{-2.5}^x$ or $h_{-4.5}^x$

[Figure 3 here.]

When s is a negative integer, an h_s^x function can be represented by an infinite number of curves. Each curve corresponds to a particular number of summation terms t that are used to compute $\sum_{s=0}^{\infty} h_s^{-x}$ for h_{-1}^x in (18). As explained in Section 4.3, each curve refers to the same h_s^x function but with different levels of computational accuracy. Figure 3 presents four indifferent curves of h_s^x with different levels of precision for one (10^2), two (10^3), three (10^4), and four (10^5) decimal places. Each set of parentheses contains the number of summation terms t for each curve.

[Figure 4 here.]

Maintaining a constant precision level, we can also plot h_s^x when s is a negative integer. As Figure 4 shows, when the precision level is set to three decimal digits ($t = 10^4$), the "h" function monotonically decreases if s belongs to non-negative integers or odd negative integers, and it monotonically increases if s belongs to even negative integers. We should be aware that only h_{-1}^x has a singular point $x = 0$, but the limit of the remaining "h" functions does exist and can be evaluated by $1/(s+1)$.

Given that the functional value of "h" and the power parameter c have a one-to-one relationship, the inverse function of "h", denoted by ${}^{-1}h_s(\delta)$, can be defined by the following infinite series and is very similar to a logarithmic function:

$$
\begin{aligned}
{}^{-1}h_s(\delta) = & -\left[(s+1)(s+2)\delta - (s+2)\right] \\
& + \frac{1}{(s+3)}\left[(s+1)(s+2)\delta - (s+2)\right]^2 \\
& - \frac{s+5}{(s+3)^2(s+4)}\left[(s+1)(s+2)\delta - (s+2)\right]^3 \\
& + \frac{s^2+11s+34}{(s+3)^3(s+4)(s+5)}\left[(s+1)(s+2)\delta - (s+2)\right]^4 + \cdots ,
\end{aligned}
\tag{19}
$$

where $\delta = h_s^c$ and $c = {}^{-1}h_s(\delta)$. To avoid confusing the inverse sign with the power parameter $c = -1$, we label the inverse function with a left superscript instead of a conventional right superscript. The formula $c = {}^{-1}h_s\left(\delta\right)$ in (19) can easily be derived from (4), and we display the first four terms here. Unfortunately, it usually requires many terms to approximate c, and it is more efficient to compute ${}^{-1}h_s\left(\delta\right)$ with iterative methods by numerical analysis. For instance, if we know $h_1^c = \delta$, the relationship between c and δ can be specified by (18) and reduced to the equation $c^2\delta - c - \exp(-c) + 1 = 0$. Solving the equation through a numerical analysis will find the root of c, or the value of the inverse function ${}^{-1}h_s\left(\delta\right)$.

[Figure 5 here.]

Figure 5 presents functional plots of $-ln(x)$ and ${}^{-1}h_s\left(x\right)$. Both functions have a similarly shaped curves, but $-ln(x)$ intersects with all the curves of ${}^{-1}h_s\left(x\right)$. Apparently, $-ln(x)$ and ${}^{-1}h_s\left(x\right)$ are distinct functions, too.

As a basic function, the "h" function has many interesting algebraic properties. The following are arithmetic rules, including addition, subtraction, multiplication, and division.

$$h_s^{p+q} = \sum_{i=0}^{\infty} \frac{q^i}{i!} \frac{\partial^{(i)} h_s^p}{\partial p^i}.$$

$$h_s^{p-q} = \sum_{i=0}^{\infty} \frac{\left(-q\right)^i}{i!} \frac{\partial^{(i)} h_s^p}{\partial p^i}.$$

$$h_s^{pq} = h_s^p + p\left(1-q\right) \sum_{i=0}^{\infty} \frac{\left(-pq\right)^i}{\prod_{j=0}^{i}\left(s+1+i\right)} h_{s+1+i}^p.$$

$$h_s^{pq^{-1}} = h_s^p + p\left(1-q^{-1}\right) \sum_{i=0}^{\infty} \frac{\left(-pq^{-1}\right)^i}{\prod_{j=0}^{i}\left(s+1+i\right)} h_{s+1+i}^p.$$

In the above formulas, p and q are interchangeable when performing addition and multiplication. Moreover, the first partial derivative of h_s^c with resepct to c is

$$\frac{\partial\left(h_s^c\right)}{\partial c} = \frac{-1}{\left(s+1\right)\left(s+2\right)} + \frac{-2(-c)^1}{\left(s+1\right)\left(s+2\right)\left(s+3\right)} + \cdots,$$

and it is equivalent to the first forward difference on s

$$\frac{\partial\left(h_s^c\right)}{\partial c}=h_{s+1}^c-h_s^c. \tag{20}$$

Taking the nth partial derivative with respect to c, the result is the nth forward difference on s

$$\frac{\partial^n\left(h_s^c\right)}{\partial c^n}=\Delta_1^n\left[h^c\right](s).$$

We derive an immediate result

$$\frac{\partial^n\left(\exp(c)h_s^c\right)}{\partial c^n}=\exp(c)h_{s+n}^c.$$

Applying (20) repeatedly, we can solve the first-order antiderivative with respect to c as shown in (21). Without loss of generality, I have not included the constant in the presentation of antiderivatives throughout this paper.

$$\int h_s^c dc=-\sum_{i=0}^{\infty}h_{s+i}^c. \tag{21}$$

The nth-order antiderivative with respect to c is

$$\int^{(n)} h_s^c dc=(-1)^n\sum_{i=0}^{\infty}\binom{i+n-1}{i}h_{s+i}^c.$$

Similarly,

$$\exp(c)h_{s-n}^c=\int^{(n)}\exp(c)h_s^c dc.$$

The result in (21) provides a key to solving the exponential integral. First, we take the Taylor expansion of $\exp(u)$ and integrate each term individually,

$$\int\frac{\exp(u)}{u}du=\log|u|+\frac{u^1}{1!(1)}+\frac{u^2}{2!(2)}+\frac{u^3}{3!(3)}+\cdots.$$

Next, let

$$y=\frac{u^1}{1!(1)}+\frac{u^2}{2!(2)}+\frac{u^3}{3!(3)}+\cdots,$$

and the first derivative of y with respect to u is h_0^{-u}. Then, y can be derived from integrating h_0^{-u} by u. According to (21), $y = \sum\limits_{s=0}^{\infty} h_s^{-u}$, and we solve the antiderivative of the exponential function as

$$\int \frac{\exp(u)}{u} du = \log|u| + \sum_{s=0}^{\infty} h_s^{-u}. \tag{22}$$

As (5) indicates, the analytical solution of the exponential integral can be expressed as $\exp(u)h_{-1}^{u}$. From (22), we can derive the definition of h_{-1}^{c} as (18) shows.

A useful identity that links h_s^c and h_{s+1}^c can be expressed as

$$(s+1)\, h_s^c = 1 - c h_{s+1}^c.$$

We can use this identity to deduce h_s^c when s is a negative integer smaller than -1. For example, let $s = -2$, and we derive $h_{-2}^c = -1 + c h_{-1}^c$. Repeat the same identity by assuming $s = -3$, $s = -4$, and so on. The result completes the definition of h_s^c for any real number s and c in (18).

In addition, the identity formula (5) makes both of the derivative and integral operators reversible:

$$\frac{\partial\left(x^s \exp(-x)\right)}{\partial x} = s x^{s-1} \exp(-x) - x^s \exp(-x),$$
$$\int \left(s x^{s-1} \exp(-x) - x^s \exp(-x)\right) dx = x^s \exp(-x),$$

where $s h_{s-1}^{-x} - x h_s^{-x} = 1$.

The nth-order partial derivative with respect to s can be generalized as

$$\frac{\partial^{(n)}\left(h_s^c\right)}{\partial s^n} = \exp(-c) \sum_{i=0}^{\infty} \frac{(-1)^n n! c^i}{i!(s+1+i)^{n+1}}.$$

Meanwhile, the nth order antiderivative with respect to s can be expressed

$$\int^{(n)} h_s^c ds = \frac{-1}{(n-1)!} \sum_{i=1}^{n-1} \binom{n-1}{n-i} s^{(n-i)} w_i \left(\sum_{j=i}^{n-1} \frac{1}{j} \right) + \frac{\exp(-c)}{(n-1)!} \left(\sum_{i=0}^{\infty} \frac{c^i}{i!} (s+1+i)^{n-1} \ln(s+1+i) \right),$$

where $w_1 = 1$, $w_2 = cw_1 + \frac{d}{dc}(cw_1)$, $\cdots$, $w_n = cw_{n-1} + \frac{d}{dc}(cw_{n-1})$.

4 Application to Major Distribution Functions

An important application of the "h" function is to unify many distribution functions that do not have a closed-form expression. In this section, the author presents an extensive study that applies "h" to explicitly specify and evaluate those functions. The author starts by discussing the question of numerical precision, and then provides a series of applications to various families of distributions, including those related to the gamma function, the exponential integral function, the error function, the beta function, the hypergeometric function, the Marcum Q-function, and the truncated normal distribution.

4.1 The finite property of c

Our previous discussion has shown that the power parameter c of the "h" function has a finite property even if it approaches infinity. Theoretically, we can compute the gamma function with arbitrary precision by increasing the value of c, but this result will not be exact unless the base parameter s is a positive integer or half-integer. Both cases are exceptional since $\Gamma(1)$ and $\Gamma\left(\frac{1}{2}\right)$ are the only cases, among all of $\Gamma(s)$ in which $s \in (0, 1]$, for which the result can be exactly proved. The idea of the factorial product, therefore, can be applied to the natural numbers and half integers. For the remaining cases, we need a numerical analysis to estimate the degree of error resulting from the choice of c.

Computationally, the finite property of c means that we can bring a considerably larger number C to replace the infinite integral limit. But how large a C qualifies as "considerably larger"? We investigate this question by evaluating numerical precision of the "h" factorization methods. Our target of analysis is $\Gamma(s)$, where $s = 0.1, 0.2, \cdots, 1$. Starting from $C = 10$, we increase C by 10 until 100 and round the result to significant decimal places. As Table 1 shows, all numerical errors pass the default precision level in Matlab's environment (15 digits) when C is 40. As we continue increasing the value of C, the precision level increases in a linear fashion. Generally, each increase of C by 10 will elevate the precision level by four decimal places. When C is set at 100, the precision level reaches 43 or more significant decimal places. In addition, the precision level also differ slightly when we vary the input argument of $\Gamma(s)$ from 0.1 to 1. Apparently, the precision level is higher when the base parameter s is closer to 0, and in other instances, it is slightly lower.

For a practical standard, C can be set to the default precision level of a user's computing environment. The author suggests $C = 20$ (at least 8 significant digits) should serve the computational purpose well. Nonetheless, since the "h" factorization method can provide a result with arbitrary precision, readers can determine their own setting of C according to the precision level they wish to achieve.

[Table 1 here.]

4.2 The Gamma function

The "h" formulas presented in (5) and (15) are the identities corresponding to the lower incomplete gamma $\gamma(s, x)$ and complete gamma function $\Gamma(s)$, respectively. We can easily transform $\gamma(s, x)$ and $\Gamma(s)$ into an "h" function. Similarly, we can also specify the upper incomplete gamma $\Gamma(s, x)$ by the difference of the complete and lower incomplete gamma functions in a form of the "h" function

$$\Gamma(s) = C^s \exp(-C) h_{s-1}^{-C},$$
$$\gamma(s,x) = x^s \exp(-x) h_{s-1}^{-x},$$
$$\Gamma(s,x) = C^s \exp(-C) h_{s-1}^{-C} - x^s \exp(-x) h_{s-1}^{-x}.$$

As noted, C refers to a considerably larger number based on the discussion in section 4.1. The ratio of the lower incomplete to complete gamma functions is defined as the regularized gamma function

$$P(s,x) = \left(\frac{x}{C}\right)^s \left(\frac{\exp(C-x) h_{s-1}^{-x}}{h_{s-1}^{-C}}\right).$$

We identify ten cumulative distribution functions that are associated with the gamma function. All of these functions can be expressed in a form of the "h" function and evaluated with arbitrary precision. Due to length restriction, we present all "h" formulas in the supplementary material, including the gamma, Poisson, chi-sqaure distributions, Erlang, inverse-gamma, chi, noncentral chi, inverse chi-square, scaled inverse chi-square, and generalized normal distributions.

4.3 The exponential integral function

The exponential integral is seldom referred to as a distribution (Meijer & Baken, 1987). However, it is a function crucial in physics and engineering science, particularly in thermal engineering (Biegen & Czanderna, 1972), hydrogeology (Barry *et al.*, 2000), stellar atmosphere (Mihalas, 2006), and petroleum engineering (Donnez, 2007). The simplest version of the exponential integral is defined as

$$E_{(1)}(x) = \int_1^\infty \frac{\exp(-xt)}{t} dt.$$

It can be transformed into a general form of the gamma integral

$$\begin{aligned} E_{(1)}(x) &= \int_x^\infty t^{-1}\exp(-t)dt \\ &= \exp(-c)h_{-1}^{-c}\Big|_{c\to\infty} - \exp(-x)h_{-1}^{-x}. \end{aligned}$$

Similarly, the nth-order exponential integral can be expressed as an upper incomplete gamma function, where the base parameter belongs to negative integers

$$\begin{aligned} E_{(n)}(x) &= x^{n-1}\int_x^\infty t^{-n}\exp(-t)dt \\ &= \left(\frac{x}{c}\right)^{n-1}\exp(-c)h_{-n}^{-c}\Big|_{c\to\infty} - \exp(-x)h_{-n}^{-x}. \end{aligned}$$

[Figure 6 here.]

We need to estimate two sources of error when applying the "h" factorization method. The first is associated with the power parameter c. As Figure 6 shows, given a considerably larger number C, $E_{(1)}$ has the largest error of all $E_{(n)}$. The upper limit of the error can thus be evaluated with the integral

$$\int_C^\infty t^{-1}\exp(-t)dt. \tag{23}$$

The result indicates that the errors are $4.16e-06$, $9.84e-11$, $8.05e-15$ for $C = 10$, 20, and 30, respectively, if we use 10^6 as the operational definition of infinity.

In addition to the error that results from C, we need to consider another source of error that is associated with the "h" function when the base argument s is a negative integer. As (18) illustrates, all of $h^c_{s,s\in\mathbb{Z}^-}$ can be reduced to a function of h^c_{-1}, which requires summing an infinite number of $h^{-c}_{s,s\in\mathbb{Z}_0^+}$. For any given $-c$, $h^{-c}_{s,s\in\mathbb{Z}_0^+}$ will slowly converge to zero if $s\to\infty$. Therefore, for computational purposes, we must determine a definitive number of the terms t to calculate this infinite series, and this decision is bound to generate a certain level of error. Notice that the exponential integrals after transformation can all be respecified as an improper integral. To evaluate this source of error, we can calculate the sum of the omitted terms under a given value C. Since $E_{(1)}$ has the heaviest tail, we use it to evaluate the upper limit of this error by

$$E_{(1)}\left(x; C, t\right) - E_{(1)}\left(x; C\right) = \lim_{n\to\infty} \sum_{s=t+1}^{n} \left(h_s^x - h_s^C\right),$$

where t represents the number of summation terms.

If we set $x = 0.5$, $C = 10$ and use 10^6 as the operational definition of infinity, the errors for $t = 10^2$, 10^3, 10^4, 10^5 are $8.87e{-}02$, $9.42e{-}03$, $9.40e{-}04$, and $8.64e - 05$, respectively. This indicates the level of precision is $m - 1$ decimal places for using 10^m terms to compute h_{-1}^{-C}. Comparing this result with the first error related to C, we conclude that the second source of error is more restrictive, since we need $t = 10^6$ terms in summing the series to reach 5 significant digits, equivalent to the maximal level of precision predicted by (23) for $C = 10$ without any restriction on t.

The above discussion reflects that the speed of convergence is a dominant factor that limits the overall level of precision. As (24) shows, the computation of $E_{(1)}\left(x; C, t\right)$ is composed of a baseline estimate $\log|C/x|$ and t negative subtracting terms $\sum_{s=0}^{t} \left(h_s^C - h_s^x\right)$, since $h_s^C < h_s^x$ for $C > 0$ and $x > 0$. While a larger C can increase precision level by reducing the first source of error, doing so biases the baseline estimate upward, and thus requires more items to achieve convergence

$$E_{(1)}\left(x; C, t\right) = \log\left|\frac{C}{x}\right| + \sum_{s=0}^{t} \left(h_s^C - h_s^x\right). \tag{24}$$

For instance, if we use the upper limit C and t summation terms to compute $E_{(1)}\left(x\right)$, the error k_1 can be specified as

$$\begin{aligned} k_1 &= E_{(1)}\left(x; C, t\right) - E_{(1)}\left(x\right) \\ &= \log(C) - \log(x) + \sum_{s=0}^{t} \left(h_s^C - h_s^x\right) - E_{(1)}\left(x\right). \end{aligned}$$

Provided that we increase the upper limit to λC ($\lambda > 1$), the new error k_2 with t summation terms will be larger than k_1

$$\begin{aligned} k_2 >& k_1 + (\lambda - 1)\exp(-C\lambda)\left[1 + (\lambda - 1)(C - 1)\, h_1^{-C(\lambda-1)}\right] \\ &+ \sum_{i=0}^{\infty} [C(\lambda - 1)]^{i+1} \exp(-C(\lambda - 1)) h_{t+1+i}^{C} h_i^{-C(\lambda-1)}, \end{aligned} \tag{25}$$

where all the three terms in the RHS equation are positive. The proof can be found in Appendix A.5.

This means that, despite the fact that increasing C can improve computational precision, the cost is slower convergence. With a given computational capacity, a tradeoff has to be made between the two sources of error for computing $E_{(n)}(x)$ efficiently. To evaluate this problem, we carry out a simulation study to see how the two sources of error interact under different settings of C and t. The precision level is set $C = 5, 10, 15, 20$, and the target of analysis includes three definite exponential integrals $E_{(1)}(x)$, $E_{(5)}(x)$, and $E_{(10)}(x)$, in which the lower limit x is set 0.5. For each definite integral, we compute the result by changing the number of summation terms from 10^3, 10^4, 10^5 to 10^6. The overall analysis contains 48 trials of numerical computations.

The simulation result in Table 2 confirms that the number of summation terms t plays a dominant role in determining the precision level of numerical computation, especially in lower-order cases. For instance, when evaluating $E_{(1)}(0.5)$ with varying upper limits C, we found that $C = 5$ and $C = 10$ has the best precision estimate under $t = 10^3, 10^4$ and $t = 10^5, 10^6$, respectively. The actual precision level fits the rule of $m - 1$ digits with 10^m summation terms very well. For higher-order cases, the precision level improves very quickly and only 10^3 terms are needed to achieve 4 and 10 significant digits under $C = 5$ for $E_{(5)}(0.5)$ and $E_{(10)}(0.5)$. The overall result indicates that a considerably larger C can be as moderate as 5 or 10, and the level of numerical precision is largely determined by the number of summation items t.

[Table 2 here.]

4.4 The error function

Through the transformation of variables, we can specify the error function, the complementary error function, and the cumulative distribution function of the standard normal distribution into an "h" function

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{x}{\sqrt{\pi}} \exp(-x^2) h_{\frac{-1}{2}}^{-x^2}, \\
\operatorname{erfc}(x) &= 1 - \frac{x}{\sqrt{\pi}} \exp(-x^2) h_{\frac{-1}{2}}^{-x^2}, \\
\Phi(x) &= \frac{1}{2}\left[1 + \frac{x}{\sqrt{2\pi}} \exp(\tfrac{-x^2}{2}) h_{\frac{-1}{2}}^{\frac{-x^2}{2}}\right].
\end{aligned}$$

Since the improper integral of the Gaussian function can be exactly analyzed, numerical error resulted from the choice of C does not apply to the three error functions. As the most widely used distribution, the error function is encountered when we integrate the probability of the normal distribution. We present the "h" formulas for eight related distribution functions in the supplementary material, including the normal, inverse Gaussian, log-normal, logit-normal, half-normal, folded normal, Maxwell-Boltzmann, and Lévy distributions.

4.5 The Beta function

The general form of the beta function is defined by the following definite integral

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt,$$

Where $\alpha > 0$ and $\beta > 0$. When the upper limit of the integral x is 1, it is the complete beta function. Otherwise, it is the incomplete beta function. The ratio of the incomplete to complete beta function is the regularized beta function, which serves as the cumulative distribution function of many distributions, such as binomial distribution, beta distribution, and F distribution.

The complete beta is closely associated with the gamma function. With some manipulations, we can transform the complete, incomplete, and regularized beta functions into a form of the "h" function

$$B\left(\alpha,\beta\right)=\frac{\exp(-C)h_{\alpha-1}^{-C}h_{\beta-1}^{-C}}{h_{\alpha+\beta-1}^{-C}},$$

$$B_x\left(\alpha,\beta\right)=x^{\alpha}\exp(-x)h_{\alpha-1}^{-x}+\sum_{i=1}^{\infty}\Bigg\{\left[\prod_{j=1,j\neq i}^{i+1}\left(\beta-j\right)\right]\left[x^{\alpha}\exp(-x)h_{\alpha-1}^{-x}-\sum_{k=0}^{i-1}(-1)^{k}\frac{x^{\alpha+k}}{k!\left(\alpha+k\right)}\right]\Bigg\},\tag{26}$$

$$I_x\left(\alpha,\beta\right)=\frac{B_x\left(\alpha,\beta\right)}{B\left(\alpha,\beta\right)}.$$

The proof for the incomplete beta function can be found in Appendix A.6.

In the supplementary material, we present the distribution functions of eight related distributions as an "h" function, such as the beta, binomial, F, beta-prime, negative binomial, Yule-Simon, noncentral F, and noncentral t distributions.

4.6 The hypergeometric function

The hypergeometric function is the core of the cumulative distribution function of Student's t distribution

$$F\left(x;\nu\right)=\frac{1}{2}+x\Gamma\left(\frac{\nu+1}{2}\right)\cdot\frac{{}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2},\frac{-x^2}{\nu}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)},$$

where $\nu>0$ (degree of freedom) and $x\in(-\infty,\infty)$ (Johnson & Kotz, 1970, p.96).

With a few steps, we can work out ${}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2},\frac{-x^2}{\nu}\right)$ in the following form:

$$
{}_2F_1\left(\tfrac{1}{2}, \tfrac{\nu+1}{2}; \tfrac{3}{2}; \tfrac{-x^2}{\nu}\right) = \frac{1}{2\sqrt{\frac{x^2}{\nu}}}\Bigg\{\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} +
$$

$$
\left(\frac{v-1}{2}\right)\left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} - \frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}}}{0!\frac{1}{2}}\right] +
$$

$$
\sum_{i=1}^{\infty}\Bigg\{\left(\frac{v-1}{2}+i\right)^2 \cdot \prod_{j=1}^{i-1}\left(\frac{v-1}{2}+j\right)
$$

$$
\left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} - \sum_{k=0}^{i}(-1)^k\frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}+k}}{k!\left(\frac{1}{2}+k\right)}\right]\Bigg\}\Bigg\}.
$$

This result demonstrates that the cumulative distribution of Student's t distribution can be expressed as an "h" function.

4.7 The Marcum Q-function

The cumulative function of the noncentral chi-square function has a form of the Marcum Q-function

$$
1 - Q_{\frac{k}{2}}\left(\sqrt{\lambda}, \sqrt{x}\right),
$$

where $k > 0$ (degree of freedom), $\lambda > 0$ (noncentrailty parameter), and $x \in [0, \infty)$. The Marcum Q-function (Nuttall, 1975) is defined as

$$
Q_M(a, b) = \int_b^{\infty} x\left(\frac{x}{a}\right)^{M-1}\exp\left(\frac{-\left(x^2+a^2\right)}{2}\right) I_{M-1}(ax)\, dx,
$$

where $I_{M-1}(ax)$ is the modified Bessel function of the first kind of $(M-1)$th order, and it is defined as

$$I_{M-1}(ax) = \frac{1}{\Gamma(M)}\left(\frac{ax}{2}\right)^{M-1}\sum_{i=0}^{\infty}\frac{1}{i!\prod_{j=1}^{i}(M-1+j)}\left(\frac{ax}{2}\right)^{2i}.$$

With some manipulations, we can simplify the Marcum Q-function into an "h" function

$$Q_M(a,b) = \frac{\exp\left(\frac{-a^2}{2}\right)}{2^M\Gamma(M)}\sum_{i=0}^{\infty}\left(\frac{a}{2}\right)^{2i}\frac{C^{M+i}\exp(\frac{-C}{2})h_{M-1+i}^{\frac{-C}{2}} - b^{2(M+i)}\exp(\frac{-b^2}{2})h_{M-1+i}^{\frac{-b^2}{2}}}{i!\prod_{j=1}^{i}(M-1+j)}.$$

In addition, the Marcum Q-function also applies to the cumulative distribution function of the Rice distribution as shown in the supplementary material.

4.8 The truncated normal distribution

The probability density function of the truncated normal distribution $TN(\mu,\sigma^2;a,b)$ can be specified as

$$f(x) = \frac{\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}{\int_a^b \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)dx}.$$

Applying (5), we can derive

$$\int_a^b \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)dx = \frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^b.$$

The cumulative distribution of the truncated normal distribution is

$$F(x) = \frac{\frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^x}{\frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^b}.$$

Let

$$D=\frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^b,$$

and we derive the moment-generating function

$$M_x\left(t\right)=\frac{\left[x-(\mu+\sigma^2t)\right]}{2D}\exp\left(\mu t+\frac{\sigma^2t^2}{2}-\frac{\left[x-(\mu+\sigma^2t)\right]^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2t\right)\right]^2}{2\sigma^2}}\Big|_a^b.$$

With a few operations, we can deduce the first and second moment of the truncated normal distribution

$$\begin{aligned}
m_1&=\mu-\frac{\sigma^2}{D}\exp\left(\frac{-\left[x-\mu\right]^2}{2\sigma^2}\right)\Big|_a^b\\
m_2&=\mu^2+\sigma^2-\frac{\sigma^2}{D}\exp\left(\frac{-\left[x-\mu\right]^2}{2\sigma^2}\right)(x+\mu)\Big|_a^b.
\end{aligned}$$

The mean and variance are therefore can be concluded as

$$\begin{aligned}
&E\left(x\right)=\mu-\frac{\sigma^2}{D}\exp\left(\frac{-\left[x-\mu\right]^2}{2\sigma^2}\right)\Big|_a^b\\
&V\left(x\right)=\sigma^2-\frac{\sigma^2}{D}\exp\left(\frac{-\left[x-\mu\right]^2}{2\sigma^2}\right)(x-\mu)\Big|_a^b-\left\{\frac{-\sigma^2}{D}\exp\left(\frac{-\left[x-\mu\right]^2}{2\sigma^2}\right)\Big|_a^b\right\}^2.
\end{aligned}$$

5 Conclusion

The author proposes an explanation based on the "h" factorization method for why the series solution to the gamma integral with a Taylor or hypergeometric function is not widely recognized. Failure to recognize the finite property of the infinite upper limit of the gamma integral results in a false conclusion that the series solution is indeterminate. Applying the "h" factorization method, the author has demonstrated, for any arbitrarily large limit c, that an even larger number N always exists, and it defines the number of expansions in the series solution, making the "h" function convergent to a value that is associated with the gamma function. This finding not only defines the algebraic meaning of the "h" function, but also explicates its elementary quality in generalizing the factorial function to the non-integer domain.

The author further studies the basic properties of the "h" function and extends its algebraic definition to all real domains for the base parameter s. For most of the core functions that are used in statistical distributions, the "h" function can serve as the minimal denominator and fully specify them exclusively with its own form. Applying this property to those cumulative distribution functions that lack closed-form expressions, we can explicitly specify their functional forms and evaluate them with arbitrary precision. These core functions include the gamma function, the exponential integral function, the error function, the beta function, the hypergeometric function, the Marcum Q-function, and the truncated normal distribution. They cover a full range of the commonly-used distributions. In addition, the "h" function can be also applied to the moment-generating function of the truncated normal distribution.

Most of the contemporary mathematicians have reached a consensus that the integrals, such as the gamma, exponential, and Gaussian integrals have no closed-form solutions, and they do not even consider it an unsolved problem. However, regardless of whether the elementary nature of the "h" function is recognized, this paper has demonstrated its application in giving the most general solution to those unsolvable integrals. Given the power of many sophisticated numerical methods today, the main contribution of the "h" function might not be computational efficiency, but rather analytical clarity of mathematical deductions, which leads to the discovery of great unity among many seemingly-unrelated distributions.

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Supplementary Material

In the supplementary material, the author includes the detailed proofs of all the formulas in the main text, the specifications of the cumulative distribution functions in "h" form, and the matlab programs for validations and replications.

Appendices

Appendix.1 Proof of (7)

We can construct the following identity by (5)

$$c^{k+1-r}\exp(c)h_{k-r}^{c} = g\left(k+1-r,1,c\right).$$

Thus, it is easy to prove (7) with two steps

$$\begin{aligned} c^{k+1-r}\exp(c)h_{k-r}^{c} &= \frac{c^{k+1-r}}{0!\left(k+1-r\right)} + \frac{c^{k+2-r}}{1!\left(k+2-r\right)} + \frac{c^{k+3-r}}{2!\left(k+3-r\right)} + \cdots, \\ h_{k-r}^{c} &= \exp(-c)\left\{\frac{c^{0}}{0!\left(k+1-r\right)} + \frac{c^{1}}{1!\left(k+2-r\right)} + \frac{c^{2}}{2!\left(k+3-r\right)} + \cdots\right\}. \end{aligned}$$

$\square$

Appendix.2 Proof of (10)

Using integration by part, we can derive $I^{(1)}\left(c,k\right)$ and $I^{(2)}\left(c,k\right)$ as

$$I^{(1)}\left(c,k\right) = \frac{I^{(0)}\left(c,k\right)}{0!(k+1)} - \frac{I^{(0)}\left(c,k+1\right)}{0!\left(k+1\right)\left(k+2\right)} + \frac{I^{(0)}\left(c,k+2\right)}{0!\left(k+1\right)\left(k+2\right)\left(k+3\right)} - \cdots,$$

$$
\begin{aligned}
I^{(2)}(c,k) = \frac{1}{1!(k+1)^2} I^{(0)}(c,k) - \Bigg[& \frac{1}{1!(k+1)^2(k+2)} \\
& + \frac{1}{1!(k+1)(k+2)^2} \Bigg] I^{(0)}(c,k+1) + \cdots .
\end{aligned}
$$

Repeating the same operations for n times, we derive

$$
\begin{aligned}
I^{(n)}(c,k) = & \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{(n-1)}}{dk} \left[\frac{1}{(k+1)} \right] I^{(0)}(c,k) \\
& + \frac{(-1)^{n}}{(n-1)!} \frac{d^{(n-1)}}{dk} \left[\frac{1}{(k+1)(k+2)} \right] I^{(0)}(c,k+1) \\
& + \frac{(-1)^{n+1}}{(n-1)!} \frac{d^{(n-1)}}{dk} \left[\frac{1}{(k+1)(k+2)(k+3)} \right] I^{(0)}(c,k+2) \\
& + \cdots .
\end{aligned}
$$

This result concludes the proof. □

Appendix.3 Proof of (11)

Given (9) and (10), we know

$$
\begin{aligned}
h^{c}_{k-r} = c^{-k-1}\exp(-c) \Bigg\{ & r^0 I^{(0)}(c,k) + r^1 \sum_{i=0}^{\infty} \left[\frac{(-1)^i}{0!} \frac{d^{(0)}}{dk} \left(\frac{1}{\prod_{j=0}^{i} (k+1+j)} \right) I^{(0)}(c,k+i) \right] \\
& + r^2 \sum_{i=0}^{\infty} \left[\frac{(-1)^{i+1}}{(1)!} \frac{d^{(1)}}{dk} \left(\frac{1}{\prod_{j=0}^{i} (k+1+j)} \right) I^{(0)}(c,k+i) \right] + \cdots \Bigg\} .
\end{aligned}
$$

Sum the series items by i, we derive

$$
\begin{aligned}
h_{k-r}^{c} =& c^{-k-1}\exp(-c)I^{(0)}\left(c,k\right) + c^{-k-1}\exp(-c)r\left\{I^{(0)}\left(c,k\right)\left(\frac{1}{0!\left(k+1-r\right)}\right)\right. \\
&-I^{(0)}\left(c,k+1\right)\left(\frac{1}{1!\left(k+1-r\right)}-\frac{1}{1!\left(k+2-r\right)}\right) \\
&\left.+\,I^{(0)}\left(c,k+2\right)\left(\frac{1}{2!\left(k+1-r\right)}-\frac{2}{2!\left(k+2-r\right)}+\frac{1}{2!\left(k+3-r\right)}\right)+\cdots\right\}.
\end{aligned}
\tag{A-1}
$$

The series shown in (A-1) is exactly the same with (11), and thus completes the proof. □

Appendix.4 Proof of (14)

To prove (14), we need to work out $\sum\limits_{i=0}^{\infty} w_i\beta_i$.

$$
\begin{aligned}
\sum_{i=0}^{n} w_i\beta_i &= \sum_{i=0}^{n}\left\{\left(\frac{1}{0!}-\sum_{j=0}^{i}\frac{c^j\exp(-c)}{j!}\right)\left(\frac{i!}{\prod\limits_{j=0}^{i}\left(-r+1+j\right)}\right)\right\} \\
&= \frac{c^1}{1!\left(-r+1\right)}-\frac{c^2}{2!\left(-r+2\right)}+\frac{c^3}{3!\left(-r+3\right)}-\frac{c^4}{4!\left(-r+4\right)}+\cdots \\
&= -c^r\Gamma\left(-r\right)-\frac{1}{r}
\end{aligned}
\tag{A-2}
$$

Bringing (A-2) back to (13) concludes the proof. See **P12** in the supplementary material for the detail. □

Appendix.5 Proof of (25)

$$
\begin{aligned}
k_2 &= E_{(1)}\left(x;\lambda C,t\right) - E_{(1)}\left(x\right) \qquad \text{(A-3)}\\
&= \log\lambda + \log C - \log x + \sum_{s=0}^{t}\left(h_s^C - h_s^x\right) + \sum_{s=0}^{t}\left(\sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i)}h_s^C}{dC^i}\right) - E_{(1)}\left(x\right)\\
&= k_1 + \log\lambda - \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_0^C}{dC^{i-1}} + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}}\\
&= k_1 + \sum_{i=0}^{\infty}\frac{C^i\exp(-C)}{i!}\left(\log\lambda + \sum_{j=1}^{i}\frac{\left(1-\lambda\right)^j}{j}\right) + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}}\\
&> k_1 + \sum_{i=0}^{\infty}\left(\frac{\left(-C\right)^i\exp(-C)}{i!}\right)\left(\frac{\left(\lambda-1\right)^{i+1}}{\left(i+1\right)} - \frac{\left(\lambda-1\right)^{i+2}}{\left(i+2\right)}\right) + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}}\\
&= k_1 + C^{-1}u\exp(-C-u)\left(h_0^{-u} - C^{-1}uh_1^{-u}\right) + \sum_{i=0}^{\infty}u^{i+1}\exp(-u)h_i^{-u}h_{t+1+i}^C\Big|_{u=C(\lambda-1)}.
\end{aligned}
$$

The last line of (A-3) is an identity of (25), and thus the proof is complete. □

Appendix.6 Proof of (26)

Gauss derived an idenity of the incomplete beta function (Dutka, 1981, p.17) as

$$
B_x\left(\alpha,\beta\right) = \frac{x^\alpha}{\alpha} - \frac{\left(\beta-1\right)x^{\alpha+1}}{1!\left(\alpha+1\right)} + \frac{\left(\beta-1\right)\left(\beta-2\right)x^{\alpha+2}}{2!\left(\alpha+2\right)} + \cdots . \qquad \text{(A-4)}
$$

We can work with (A-4) by rearranging it as an infinite series of the "h" function:

$$
\begin{aligned}
B_x(\alpha,\beta) = {} & x^{\alpha}\exp(-x)h_{\alpha-1}^{-x} + x^{\alpha}(\beta-2)\left(\exp(-x)h_{\alpha-1}^{-x} - \frac{1}{\alpha}\right) \\
& + x^{\alpha}(\beta-1)(\beta-3)\left(\exp(-x)h_{\alpha-1}^{-x} - \frac{1}{\alpha} + \frac{x^1}{1!\,(\alpha+1)}\right) \\
& + x^{\alpha}(\beta-1)(\beta-2)(\beta-4)\left(\exp(-x)h_{\alpha-1}^{-x} - \frac{1}{\alpha} + \frac{x^1}{1!\,(\alpha+1)} - \frac{x^2}{2!\,(\alpha+2)}\right) \\
& + \cdots .
\end{aligned}
$$

The result concludes the proof. □

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C	10	20	30	40	50	60	70	80	90	100
$\Gamma(0.1)$	5	9	14	18	23	27	32	36	40	45
$\Gamma(0.2)$	5	9	14	18	23	27	31	36	40	45
$\Gamma(0.3)$	5	9	14	18	22	27	31	36	40	44
$\Gamma(0.4)$	4	9	13	18	22	27	31	35	40	44
$\Gamma(0.5)$	4	9	13	18	22	26	31	35	40	44
$\Gamma(0.6)$	4	9	13	18	22	26	31	35	39	44
$\Gamma(0.7)$	4	9	13	17	22	26	30	35	39	44
$\Gamma(0.8)$	4	8	13	17	22	26	30	35	39	43
$\Gamma(0.9)$	4	8	13	17	21	26	30	34	39	43
$\Gamma(1)$	4	8	13	17	21	26	30	34	39	43

Table 1: Total Number of Significant Digits by Varying C

$E_{(n)}(x)$	t	$C = 5$	$C = 10$	$C = 15$	$C = 20$
$E_{(1)}(0.5)$	10^3	**2**	2	1	1
	10^4	**3**	3	2	2
	10^5	2	**4**	3	3
	10^6	2	**5**	4	4
$E_{(5)}(0.5)$	10^3	**4**	4	4	4
	10^4	**5**	5	5	5
	10^5	**7**	6	6	6
	10^6	7	**7**	7	7
$E_{(10)}(0.5)$	10^3	**10**	10	10	9
	10^4	**11**	11	11	10
	10^5	12	**12**	12	11
	10^6	12	**13**	13	12

Table 2: Simulation Results of Numerical Precision by Varying C and t

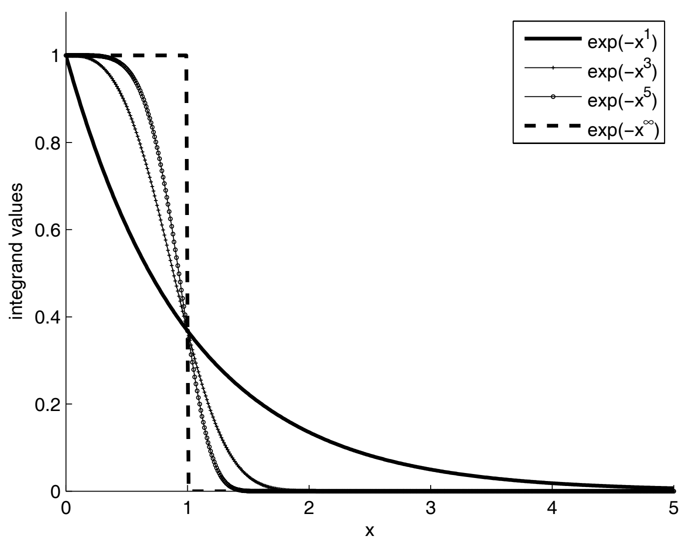

Figure 1: Functional Plots of $\exp\left(-x^{p/(p-q)}\right)$

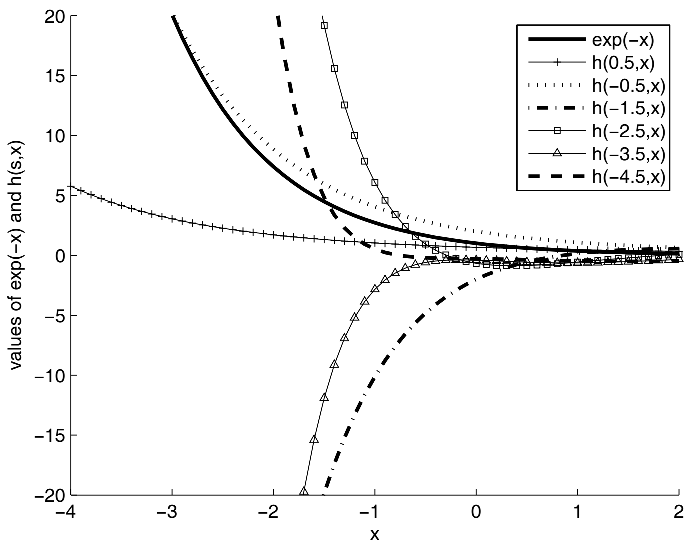

Figure 2: Functional Plots of $\exp(-x)$ and h_s^x

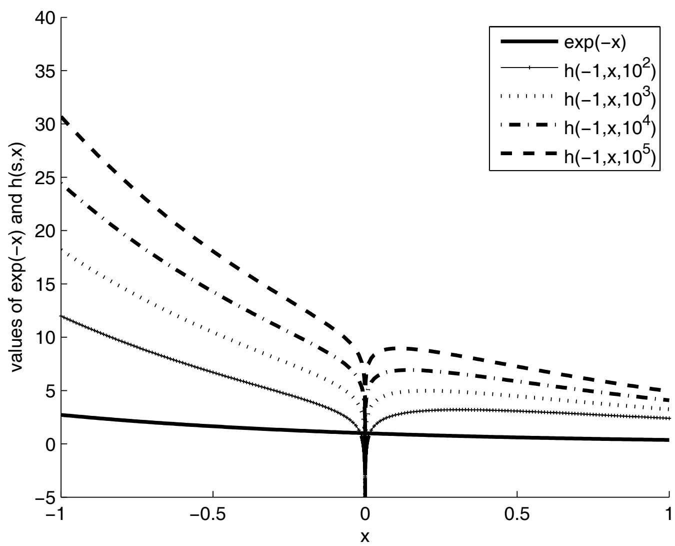

Figure 3: Functional Plots of $\exp(-x)$ and h_{-1}^{x} by varying t

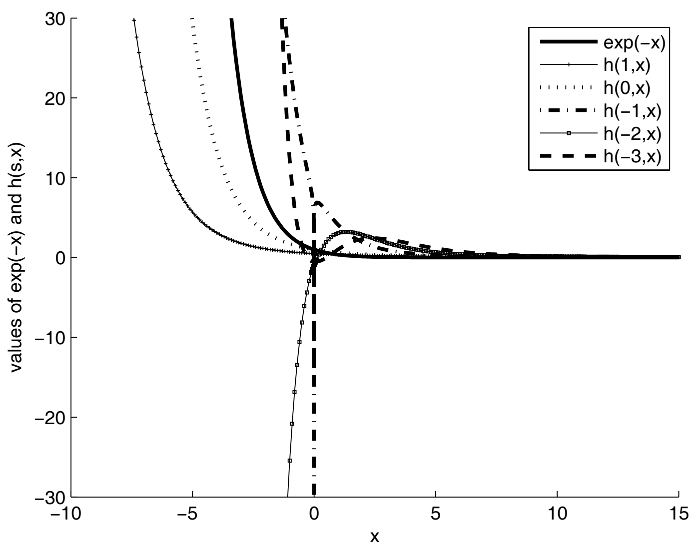

Figure 4: Functional Plots of $\exp(-x)$ and h_s^x with three significant digits (if $s \in \mathbb{Z}^-$)

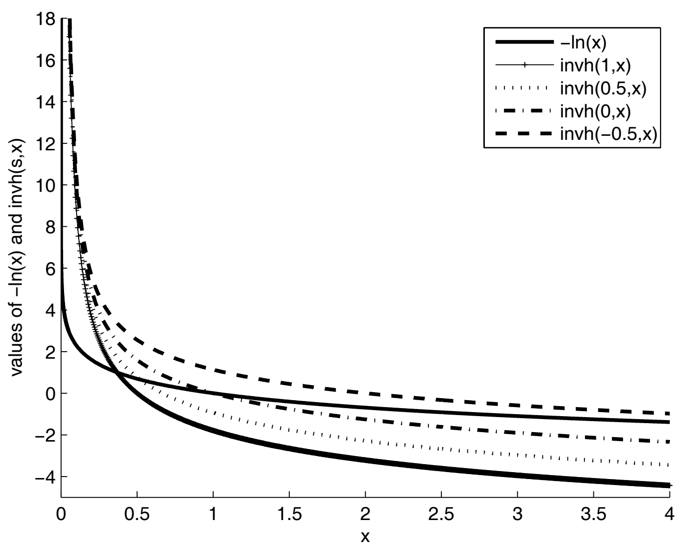

Figure 5: Functional Plots of $-ln(x)$ and ${}^{-1}h_s\,(x)$

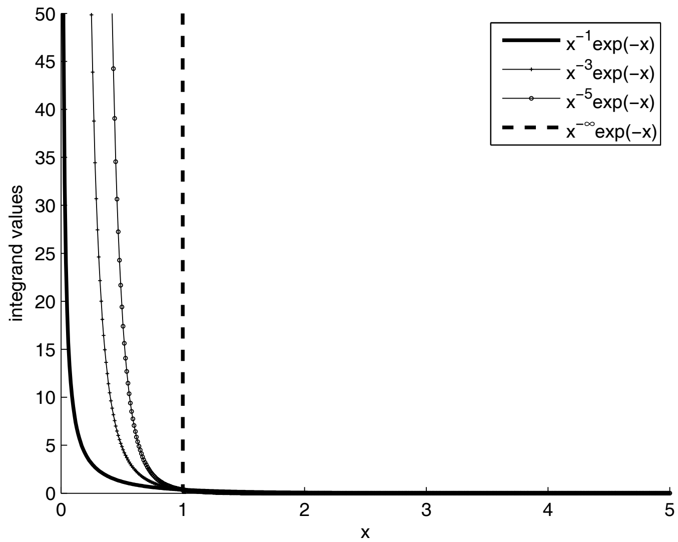

Figure 6: Functional Plots of $x^{-n}\exp(-x)$